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## Long-Term Coupling Effects between Librational and Orbital Motions of a Satellite

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### Introduction

FOR a dynamically nonspherical satellite, Newton's equations of orbital motion are coupled to Euler's equations of rigid-body rotational motion by high-order body-force terms. The body force is produced by the earth's gravity gradient and by the nonspherical inertia ellipsoid of the satellite. The body force is much smaller than the central force that acts on the satellite as though the satellite is a point mass. Their ratio is proportional to  $L^2/r^2$ , where  $L$  is a characteristic dimension of the satellite and  $r$  the geocentric distance of the center of mass of the satellite. It is through this small body force that the orbital and the rotational motions affect each other. Ordway<sup>1</sup> and Moran<sup>2</sup> have investigated the coupling effects for the case of an undamped dumbbell satellite in a circular orbit, showing that the rotational motion will produce sinusoidal perturbations to the basic orbit. In this note, however, a study is presented of the long-term coupling effects for a satellite of general shape, provided with damping, and in an elliptic orbit.

In the passive attitude stabilization of a satellite either by gravitational or by magnetic means, it is necessary that the rotational motion of the satellite be provided with damping by internal or external means. Damping by external means, as often used in magnetic orientation,<sup>3</sup> is obtained from interaction with, e.g., the geomagnetic field by a device provided in the satellite. Damping by internal means is produced by the relative motion between the satellite and an auxiliary body contained in or connected to the satellite, and the damping can be either of the viscous type or of the hysteresis type.<sup>4</sup> A question is often raised as to where the energy, which is being continuously dissipated through the damping mechanism, comes from. One can surmise that the dissipated energy originates from the initial angular momentum about the center of mass of the satellite when entering the orbit, from the sporadic and the continuous momentum inputs from space environments, and from the orbital motion. The initial and the subsequent angular momentum inputs will be dissipated into heat through the damping mechanism within a reasonably short time.<sup>4</sup> If, however, the rotational motion is excited continuously by the orbital motion (i.e., by the orbital eccentricity in the case of gravitational orientation or even by a circular orbital motion in the case of magnetic orientation), then damping of the rotational motion will result in a continuous dissipation of the orbital energy.

In this note we will investigate the nature of the long-term coupling effects between the orbital and the rotational motion in the case of gravitational orientation provided with damping by internal means. For this reason it is irrelevant to consider at the same time other effects due to, e.g., space environment, earth's oblateness, etc., despite the fact that these latter effects are most likely bigger in magnitude than those due to the rotational motion. For the same reason, the present work is restricted to the special case of planar pitch librational motion.

### Equations for Planar Pitch Motion

It is assumed that earth is a uniform sphere of radius  $R_0$  and that the satellite is a composite body of a total mass  $m$ , consisting of a main rigid body and an auxiliary rigid body, connected to each other through a hinge joint such that their centers of mass coincide. The coordinate systems are chosen such that the orbit lies in the  $XZ$  plane of a nonrotating coordinate system  $O-XYZ$  with the orbital angular momentum vector pointing in the  $Y$  direction. The origin  $O$  is situated at the geocenter. The mass center  $o$  of the satellite is specified by polar coordinates  $(r, \Theta)$ , where  $\Theta$  is the angle measured from the  $Z$  axis in the direction of the orbital motion. A rotating coordinate system  $o-x'y'z'$  is taken in such a way that  $oz'$  is parallel to  $Oo$  and  $oy'$  to  $OY$ . The body coordinate systems are chosen with the  $o-x_1y_1z_1$  and  $o-x_2y_2z_2$  axes of body 1 and 2, respectively, along the principal axes with moments of inertia  $(I_{x_i}, I_{y_i}, I_{z_i})$ ,  $i = 1, 2$ . The satellite is assumed to undergo only a pitch rotational motion such that its  $z_1$  and  $z_2$  axes make an angle  $\theta_1$  and  $\theta_2$ , respectively, with the  $z'$  axis in the orbital plane, and the  $y_1$  and  $y_2$  axes are always parallel to  $oy'$  or  $OY$ . In the hinge joint of the composite satellite, the spring torque is assumed to be linear with the relative angular displacement ( $k$  = spring constant) and the viscous damping torque to be linear with the relative angular velocity between the two bodies ( $c$  = damping coefficient). With terms of order higher than  $L^2/r^2$  neglected, the following four equations result:

$$\ddot{r} - r\dot{\Theta}^2 = -\frac{\mu}{r^2} - \frac{3}{2}\frac{\mu}{mr^4}[-3I_{x_1}\sin^2\theta_1 - 3I_{z_1}\cos^2\theta_1 + (I_{x_2} + I_{y_2} + I_{z_2}) - 3I_{x_2}\sin^2\theta_2 - 3I_{z_2}\cos^2\theta_2 + (I_{x_2} + I_{y_2} + I_{z_2})] \quad (1)$$

$$2r\ddot{\Theta} + r\dot{\Theta} = 3(\mu/mr^4)[(I_{x_1} - I_{z_1})\sin\theta_1\cos\theta_1 + (I_{x_2} - I_{z_2})\sin\theta_2\cos\theta_2] \quad (2)$$

$$I_{y_1}(\ddot{\Theta} + \ddot{\theta}_1) = -3(\mu/r^3)(I_{x_1} - I_{z_1})\sin\theta_1\cos\theta_1 - c(\dot{\theta}_1 - \dot{\theta}_2) - k(\theta_1 - \theta_2) \quad (3)$$

$$I_{y_2}(\ddot{\Theta} + \ddot{\theta}_2) = -3(\mu/r^3)(I_{x_2} - I_{z_2})\sin\theta_2\cos\theta_2 - c(\dot{\theta}_2 - \dot{\theta}_1) - k(\theta_2 - \theta_1) \quad (4)$$

where dots denote time derivatives and  $\mu = gR_0^2$ , with  $g$  being the gravitational acceleration at the earth's surface.

### Method of Successive Approximations

Equations (1-4) will be solved by a method of successive approximations. First, the body-force terms of  $O(L^2/r^2)$  are neglected in (1) and (2). The orbital equations become decoupled from the rotational equations. For chosen initial conditions, the Keplerian orbit can be immediately obtained as an ellipse of semimajor axis  $r_0$ , phase angle  $\varphi_0$ , and eccentricity  $e < 1$  described by the following two relations:  $r = r_0(1 - e^2)/[1 + e\cos(\Theta + \varphi_0)]$  and  $\dot{\Theta} = [\mu r_0(1 - e^2)]^{1/2}/r^2$ . With the aid of the foregoing relations, (3) and (4) can be solved in the librational case, the solutions of which are then in turn substituted into the orbital equations to solve for the first-order solution, or solution including terms of the same order as the body force or  $O(L^2/r^2)$ .

To linearize (3) and (4) for the case of a librational motion, we assume that  $\theta_i$  is of the order of  $e$ , which is taken to be small here as compared with unity. From the orbital relations, we have  $\dot{\theta} = \Omega + O(e)$  and  $\Theta = \Omega t + O(e)$ , where  $\Omega = (\mu/r_0^3)^{1/2}$ . With the terms of  $O(\theta_i^2, e\theta_i, e^2)$  neglected, Eqs (3) and (4) become

$$I_{y_i}\ddot{\theta}_i \pm c(\dot{\theta}_1 - \dot{\theta}_2) \pm k(\theta_1 - \theta_2) + (I_{x_i} - I_{y_i})\Omega^2\theta_i = 2I_{y_i}e\Omega^2 \sin(\Omega t + \varphi_0) \quad i = 1, 2 \quad (5)$$

The steady-state solutions of the two preceding equations are

$$\theta_i = F_i \cos(\Omega t + \varphi_0) + G_i \sin(\Omega t + \varphi_0) \quad i = 1, 2 \quad (6)$$

Here

$$\begin{aligned} F_1 &= 2e\Delta^{-1}\bar{c}_1P_2(P_1 - P_2) \\ G_1 &= 2e\Delta^{-1}[P_1P_2^2 + (\bar{k}_1 + 2\bar{k}_2)P_1P_2 + \bar{k}_1P_2^2 + \\ &\quad P_1\{\bar{k}_1\bar{k}_2 + \bar{k}_2^2 + \bar{c}_2(\bar{c}_1 + \bar{c}_2)\} + \\ &\quad P_2\{\bar{k}_1\bar{k}_2 + \bar{k}_1^2 + \bar{c}_1(\bar{c}_1 + \bar{c}_2)\}] \end{aligned}$$

$F_2$  and  $G_2$  are the same as  $F_1$  and  $G_1$ , respectively, except that the subscripts 1 and 2 are interchanged. In the preceding,  $\bar{k}_i = k/I_{y_i}\Omega^2$ ,  $\bar{c}_i = c/I_{y_i}\Omega$ ,  $P_i = 3(I_{x_i} - I_{y_i})/I_{y_i} - 1$ ,  $i = 1, 2$ , and  $\Delta = [P_1P_2 + \bar{k}_1P_2 + \bar{k}_2P_1]^2 + [\bar{c}_2P_1 + \bar{c}_1P_2]^2$ .

Combination of (2) with (3) and (4) yields

$$\dot{\Theta} = (h - I_{y_1}'\dot{\theta}_1 - I_{y_2}'\dot{\theta}_2)/(I_{y_1}' + r^2)$$

where  $I_{y_1}' = I_{y_1}/m$ ,  $I_{y_2}' = I_{y_2}/m$ ,  $I_{y_1}' = I_{y_1}' + I_{y_2}'$ , and  $h = \frac{1}{2}\dot{\theta}_0 + I_{y_1}'(\dot{\theta}_0 + \dot{\theta}_{10}) + I_{y_2}'(\dot{\theta}_0 + \dot{\theta}_{20})$ , with  $\dot{\theta}_0$ ,  $\dot{\theta}_{10}$ ,  $\dot{\theta}_{20}$ , and  $\bar{r}$  being the initial values of  $\dot{\theta}$ ,  $\dot{\theta}_1$ ,  $\dot{\theta}_2$ , and  $r$ , respectively, at  $t = 0$ . Introduce a nondimensional variable  $u^* = r_p/r$ , where  $r_p$  is the initial geocentric distance at perigee, and make  $\Theta$  the independent variable; then

$$\dot{\Theta} = u^{*2} \frac{(h - I_{y_1}'\dot{\theta}_1 - I_{y_2}'\dot{\theta}_2)}{(r_p^2 + I_{y_1}'u^{*2})} \quad \dot{r} = -\dot{\Theta} \frac{r_p}{u^{*2}} \frac{du^*}{d\Theta}$$

$$\ddot{r} = r_p \frac{du^*}{d\Theta} \frac{[I_{y_1}'\ddot{\theta}_1 + I_{y_2}'\ddot{\theta}_2 + 2I_{y_1}'\dot{\Theta}(1/u^*)(du^*/d\Theta)]}{(r_p^2 + I_{y_1}'u^{*2})} - \dot{\Theta}^2 \frac{r_p}{u^{*2}} \frac{d^2u^*}{d\Theta^2}$$

Then Eq (1) becomes

$$\begin{aligned} \frac{d^2u^*}{d\Theta^2} + u^* - \frac{\mu}{r_p^3} \frac{(r_p^2 + I_{y_1}'u^{*2})^2}{(h - I_{y_1}'\dot{\theta}_1 - I_{y_2}'\dot{\theta}_2)^2} - \\ \frac{\delta(r_p^2 + I_{y_1}'u^{*2})^2}{r_p^5(h - I_{y_1}'\dot{\theta}_1 - I_{y_2}'\dot{\theta}_2)^2} \times \\ \left\{ u^{*2} + \left[ \frac{r_p^5(I_{y_1}'\ddot{\theta}_1 + I_{y_2}'\ddot{\theta}_2)}{\delta(r_p^2 + I_{y_1}'u^{*2})u^{*2}} + \right. \right. \\ \left. \left. \frac{2I_{y_1}'r_p^5(h - I_{y_1}'\dot{\theta}_1 - I_{y_2}'\dot{\theta}_2)^2u^*}{\delta(r_p^2 + I_{y_1}'u^{*2})^3} \frac{du^*}{d\Theta} \right] \frac{du^*}{d\Theta} \right\} = 0 \quad (7) \end{aligned}$$

where  $\delta$  is the coefficient of  $-r^{-4}$  in (1). To solve for a first-order solution, let us keep  $\delta$  fixed by taking its average value in one unperturbed orbit  $\bar{\delta}$ , which is made nonzero by a proper choice of the moments of inertia and is, in general, positive. Denote  $\epsilon = \bar{\delta}/(h^2r_p) \approx L^2/r^2(\ll 1)$ ; then the third and the fourth terms in (7) can be expanded into terms of  $O(1)$ ,  $O(\epsilon)$ ,  $O(\epsilon^2)$ . By retaining terms up to  $O(\epsilon)$ , letting  $v^* = u^* - (\mu/h^2)r_p$ , and with the aid of (6), Eq (7) is brought to the form

$$\frac{d^2v^*}{d\Theta^2} + v^* - \epsilon f\left(\Theta, v^*, \frac{dv^*}{d\Theta}\right) = 0 \quad (8)$$

Here the nonlinear function is given as

$$\begin{aligned} f\left(\Theta, v^*, \frac{dv^*}{d\Theta}\right) = a(v^* + \beta)^2 + b(v^* + \beta) \left(\frac{dv^*}{d\Theta}\right)^2 + \\ \left[ d - n(v^* + \beta)^{-2} \frac{dv^*}{d\Theta} \right] \sin(\Theta + \varphi_0) + \\ \left[ p + q(v^* + \beta)^{-2} \frac{dv^*}{d\Theta} \right] \cos(\Theta + \varphi_0) \quad (9) \end{aligned}$$

where the dimensionless quantities are

$$\begin{aligned} \beta &= \frac{\mu}{h^2} r_p \left( = \frac{1}{1 + e} \right) \\ a &= 1 + \frac{2\mu I_{y_1}'}{\bar{\delta}} \quad b = \frac{2I_{y_1}'h^2}{\bar{\delta}r_p} \\ d &= \frac{4\epsilon c\mu r_p^2(P_1 - P_2)^2}{m\bar{\delta}h\Delta} \quad n = \frac{r_p^3\Omega^2(I_{y_1}'G_1 + I_{y_2}'G_2)}{\bar{\delta}} \\ p &= \frac{2r_p^2\mu\Omega(I_{y_1}'G_1 + I_{y_2}'G_2)}{\bar{\delta}h} \quad q = \frac{2\epsilon cr_p^3\Omega(P_1 - P_2)^2}{m\bar{\delta}\Delta} \end{aligned}$$

Note that we have replaced  $\Omega t$  by  $\Theta$  in the expression of  $\theta$ , since  $\theta = O(e)$ .

#### First-Order Solution by Kryloff-Bogoliuboff Method

Equation (8) represents a nonautonomous oscillation system with resonant external periodic excitation. The nonlinear forcing function  $f$  is a finite trigonometric polynomial in  $\Theta$  with coefficients being polynomials in  $v^*$  and  $dv^*/d\Theta$ . If  $v^*$  takes the form of  $v^* = A \cos\psi$ ,  $\psi = \Theta + \varphi$ , then the preceding forcing function can be developed into a double Fourier series. Therefore, it is evident from (8) that resonance occurs at the first harmonic terms, since their combination frequency is equal to the natural frequency of the system, which is unity. This is classified as the "exact" resonance according to Minorsky.<sup>5</sup> Let the first-order solution be written as

$$v^* = A(\Theta) \cos\psi = A(\Theta) \cos(\Theta + \varphi(\Theta)) \quad (10)$$

The amplitude  $A(\Theta)$  and the phase angle  $\varphi(\Theta)$  are to satisfy the following first-order differential equations<sup>6</sup>:

$$\begin{aligned} \frac{dA}{d\Theta} = -\frac{\epsilon}{2} \left\{ d \cos(\varphi_0 - \varphi) - p \sin(\varphi_0 - \varphi) - \right. \\ \left. [n \sin(\varphi_0 - \varphi) - q \cos(\varphi_0 - \varphi)] \frac{2}{A^2} \left[ \frac{2\beta^2 - A^2}{(\beta^2 - A^2)^{1/2}} - 2\beta \right] \right\} \quad (11) \end{aligned}$$

$$\begin{aligned} \frac{d\varphi}{d\Theta} = -\frac{\epsilon}{2A} \left\{ 2a\beta A + \frac{1}{4} bA^3 + d \sin(\varphi_0 - \varphi) + \right. \\ \left. p \cos(\varphi_0 - \varphi) - [n \cos(\varphi_0 - \varphi) + \right. \\ \left. q \sin(\varphi_0 - \varphi)] \frac{2}{A^2} \left[ \frac{2\beta^2 - A^2}{(\beta^2 - A^2)^{1/2}} - 2\beta \right] \right\} \quad (12) \end{aligned}$$

The preceding two nonlinear first-order equations are coupled to each other and cannot be integrated in an exact way. Yet, a first-order solution in  $\epsilon$  can be obtained. We assume that the solution for  $\varphi$  and  $A$  takes a series form, namely,  $\varphi = \varphi_0 + \epsilon\varphi_1(\Theta) + \epsilon^2\varphi_2(\Theta) + \dots$ , with  $\varphi(\Theta = -\varphi_0) = \varphi_0$  or  $\varphi_i(\Theta = -\varphi_0) = 0$ ,  $i = 1, 2, \dots$ , and  $A = A_0 + \epsilon A_1(\Theta) + \dots$ , where  $A_0 = e/(1 + e)$ , and  $A_i(\Theta = -\varphi_0) = 0$ ,  $i = 1, 2, \dots$ . Then  $A_1(\Theta)$  can be easily obtained as

$$\begin{aligned} A_1 = -2e(1 + e)(1 - e)^3 \frac{c\mu(P_1 - P_2)^2}{\Omega m\bar{\delta}\Delta} \times \\ \left[ (1 - e^2)^{-3/2} + \frac{1}{e^2} \left( \frac{2 - e^2}{(1 - e^2)^{1/2}} - 2 \right) \right] (\Theta + \varphi_0) \quad (13) \end{aligned}$$

The second term in the brackets of (13) is positive definite for  $e < 1$  and has a zero limiting value as  $e \rightarrow 0$ . In a similar way, the first-order solution of  $\varphi$  is found as  $\varphi_1 = \bar{\varphi}_1(\Theta + \varphi_0)$ , where

$$\bar{\varphi}_1 = - \left\{ \left( 1 + \frac{2\mu I_y'}{\delta} \right) \frac{1}{(1+e)} + \frac{1}{4} \frac{e^2}{(1+e)} \frac{I_y' \mu}{\delta} + \frac{1}{e} (1+e)^2 (1-e)^3 \times \right. \\ \left. (I_{y1}' G_1 + I_{y2}' G_2) \frac{\mu}{\delta} \left[ (1-e^2)^{-3/2} - \frac{1}{e^2} \left( \frac{2-e^2}{(1-e^2)^{1/2}} - 2 \right) \right] \right\} \quad (14)$$

By substituting  $A = A_0 + \epsilon A_1(\Theta)$  and  $\varphi = \varphi_0 + \epsilon \varphi_1(\Theta)$  into (10) and changing the variables back to  $r$ , the perturbed orbit resulting from the steady-state forced librational motion is obtained. Note that the initial phase angle  $\varphi_0$  can be taken to be equal to zero by transformation of the coordinates  $O-XYZ$ .

Orbital decay rate can be found from the expression of the semimajor axis  $r_0(\Theta)$  of the orbital ellipse, which is one-half the sum of the two apsidal distances, corresponding to  $\cos(1 + \epsilon \bar{\varphi}_1)\Theta = \pm 1$ , i.e.,  $r_0(\Theta) = r_0 + \Delta r_0(\Theta)$ , where

$$\Delta r_0(\Theta) = - \frac{4c(P_1 - P_2)^2}{r_0 \Omega m \Delta} e^2 \left[ (1 - e^2)^{-3/2} + \frac{1}{e^2} \left( \frac{2 - e^2}{(1 - e^2)^{1/2}} - 2 \right) \right] \quad (15)$$

Orbital phase shift is simply

$$\epsilon \varphi_1 = - \left\{ \frac{(\delta/\mu + 2I_y')}{r_0^2 (1 - e^2)^2} + \frac{e^2}{4(1 - e^2)^2} \frac{I_y'}{r_0^2} + \frac{1}{r_0^2} \left( I_{y1}' \frac{G_1}{e} + I_{y2}' \frac{G_2}{e} \right) (1 - e^2) \times \right. \\ \left. \left[ (1 - e^2)^{-3/2} - \frac{1}{e^2} \left( \frac{2 - e^2}{(1 - e^2)^{1/2}} - 2 \right) \right] \right\} (\Theta) \quad (16)$$

### Numerical Example

To illustrate numerically the magnitude of the rates of orbital decay and phase shift, let us take a two-body satellite with the following parameters, which has been demonstrated in Refs. 4 and 7 to be feasible in gravitational orientation:  $m = 10$  slugs,  $I_{x1} = I_{y1} = 3300$ ,  $I_1 = 10$ ,  $I_{x2} = 450$ ,  $I_{y2} = 1000$ ,  $I_z = 1450$  slug-ft<sup>2</sup>, coefficient of viscous friction  $c = 4270\Omega$  ft-lb-sec, spring constant  $k = 4300\Omega^2$  ft-lb/rad. Hence,  $I_y' = 430$  ft<sup>2</sup>,  $\delta/\mu \approx 775$  ft<sup>2</sup>,  $P_1 = 2$ ,  $P_2 = -4$ ,  $\bar{c}_1 = 1.28$ ,  $\bar{c}_2 = 4.27$ ,  $k_1 = 1.29$ ,  $\bar{k}_2 = 4.3$ ,  $\Delta = 32.5$ ,  $G_1 = -2.82e$ ,  $G_2 = -4.7e$ . Then, for 1000-mile alt or  $r_0 = 2.62 \times 10^7$  ft ( $\Omega = 0.89 \times 10^{-3}$  rad/sec) and  $e = 0.1$  (note: from the work of Refs. 4 and 7,  $e$  should be kept smaller than 0.15 for the librational motion to be valid), it is computed from (15) that the rate of orbital decay is  $\Delta r_0/\Delta t = 0.02$  ft/yr and from (16) that the rate of apsidal advance is  $\Delta \varphi/\Delta t = 0.65 \times 10^{-6}$  deg/yr.

The orbital decay rate can be estimated in another way from the formula  $\Delta r_0 \approx (m\mu/2)(\Delta E/E^2) = (2/gm)(r_0/R_0)^2 \Delta E$ , where  $E = -\mu m/2r_0$  is the total orbital energy, and

$$\Delta E \approx \int_0^{2\pi/\Omega} c(\dot{\theta}_1 - \dot{\theta}_2)^2 dt = \pi c \Omega [(F_1 - F_2)^2 + (G_1 - G_2)^2]$$

is the energy dissipation per cycle through the steady-state forced librational motion. From this, it is calculated that  $\Delta r_0 \approx 0.024$  ft/yr, which checks fairly well with the result given in (15). The same type of approximation applied to a magnetically oriented satellite should produce results of comparable accuracy.

From the preceding numerical results it is concluded that these long-term effects are negligibly small for a gravitationally oriented satellite. It may be of interest, however, to attempt an application of the foregoing formulas to astronomical problems.

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## Energy Solution for Simply Supported Oval Shells

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### Introduction

A RECENT study of clamped, short, oval shells under hydrostatic load<sup>1</sup> presented an energy solution of the shell equations. The present note shows the results of applying this solution to simply supported, short, oval shells under constant lateral load. The shells considered have major-to-minor axis ratios  $b/a$  in the range  $1.00$  (circular)  $\leq b/a \leq 2.06$ . For values of  $b/a > 2.06$ , the assumed shell cross section [see Eq. (1)] does not remain convex at all points.

The results are compared to a double Fourier series solution, which is presented in Refs. 2 and 3 for values of  $b/a$  of 1.10, 1.51, and 2.06 and which is herein considered to be exact. For other values of  $b/a$  an equivalent circular shell solution, which in Refs. 2 and 3 is shown to be in good agreement with the exact solution, has been used as a basis for comparison. This approximate solution is obtained by using the value of the local radius of curvature of the oval shell for the radius in the well-known axisymmetric solution of the circular, cylindrical shell equations.

The energy solution used can be applied to a shell having arbitrary boundary conditions (e.g., a shell with elastic ring supports), whereas the double Fourier series or equivalent circular shell solutions are not readily applicable to such a shell.

Received August 29, 1963; revision received December 30, 1963. This work was jointly sponsored by the Office of Naval Research and the Bureau of Ships under Contract No. Nonr 839(14), Project NR 064 167. Additional results are given in Ref. 5.

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